Développements gPC pour la propagation d'ondes en milieux aléatoires

Panorama EDF/CEA/Safran

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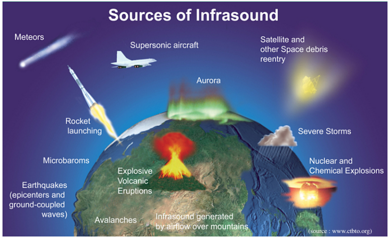
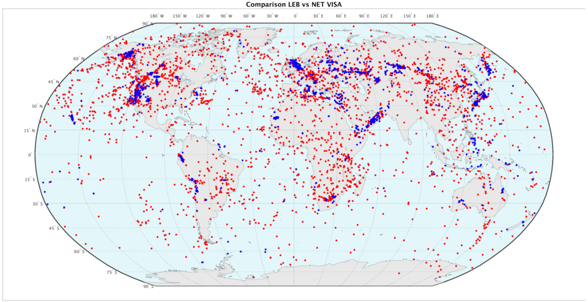
WHY ARE WE INTERESTED IN ATMOSPHERIC PROPAGATION ?

- The verification regime of the Comprehensive Nuclear-Test-Ban Treaty (CTBT) is designed to detect any nuclear explosion conducted on Earth – underground, underwater or in the atmosphere.
- Infrasound monitoring is one of the four technologies used by the International Monitoring System (IMS) to verify compliance with the CTBT.

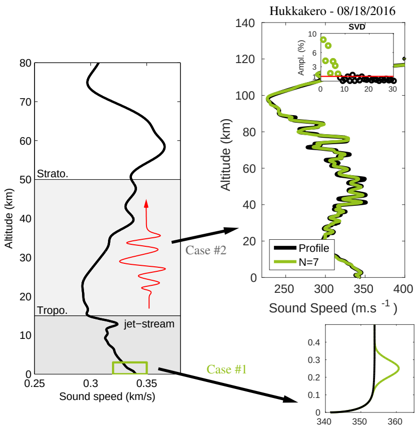




- Infrasound has the ability to cover long distances with little dissipation but signals can be severly distorted during propagation.
- Many events are detected but an important part of them does not correspond to physical events but only to propagation effects.

THE ATMOSPHERIC UNCERTAINTIES

- Atmospheric Specification (AS) :
  - Background state provided by numerical weather forecasts / atmospheric climate reanalysis.
  - Fluctuations of those profiles not resolved.
- Uncertainties associated with AS :
  - Come from complex physical phenomena (Turbulence, Gravity Waves,...)
  - Statistical study can be conducted to characterize the uncertainty and describe it by a vector $\xi \in \mathbb{R}^d$.
- Assessing the impact of ξ on infrasound :
  - Deviation from mean state (due to randomness) can be of great amplitude but the dimension is low $d < 10$.
  - Long-range propagation can be simulated accurately but with high computational cost.
  - For computational model parameters θ, $Y = F(X(\xi);\theta)$ can be represented using a metamodel.



OUTLINE

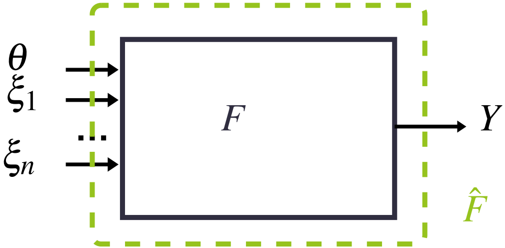
1. Polynomial Chaos based Metamodel
   - What is a Metamodel ?
   - The Polynomial Chaos (gPC) Framework
   - The problem of long-term integration
   - A quick review of the existing methods

2. Modal decomposition
   - Spectrum of the propagating operator
   - From spectral expansion to temporal signals

3. gPC & Modal decomposition
   - Modal decomposition for gPC
   - Technical point : isolate a mode
   - Different gPC for different physics

ON THE USE OF A METAMODEL

Build a metamodel $\hat{F}$ of $Y = F(X(\xi);\theta)$ where $\xi \sim \mathcal{N}(0,\mathbb{I}_n)$:



- Characteristics of a metamodel :
  - Calibrated using a small number of runs of the expensive code.
  - Easy to assess numerically.
  - Reproduce the statistical properties of the output.
- Application to infrasound propagation :
  - Forward uncertainty propagation : Y is a signal at a given distance R
  - Association or localization : Y is a set of by-products (duration, amplitude, ...)

Polynomial Chaos decomposition

- A non-intrusive metamodel of $Y = F(X(\xi);\theta)$ where $\xi \sim \mathcal{N}(0, I_n)$

- Polynomial chaos decomposition :

$$Y = \sum_{j\in J} a_j H_j(\xi) \text{ and } a_j = \langle Y, H_j \rangle,$$

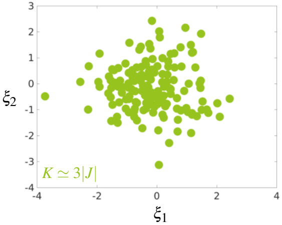
  - $(H_j)_{j\in J}$ set of polynomials (up to degree d)
  - $(H_j)_{j\in J}$ orthonormals for inner product

$$\langle f, g \rangle = \mathbb{E}[fg]$$

- Cross-validation for order ($|J|$) selection : Leave-One-Out procedure.
- Sobol' indices can be expressed with the coefficients for sensitivity analysis.
- Once the $(a_j)_{j\in J}$ are computed, Y can be easily computed by evaluating the polynomials.

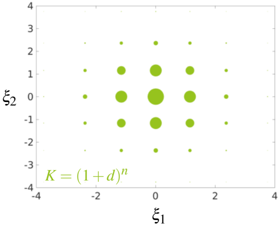
- Computation of coefficients $(a_j)_{j\in J}$

Point Cloud - Regression $w^{(k)} = 1$



$$(a_j)_{j\in J} = \underset{a\in\mathbb{R}^{|J|}}{\arg\min} \|Y(\xi) - \sum_{j\in J} a_j H_j(\xi)\|_2$$

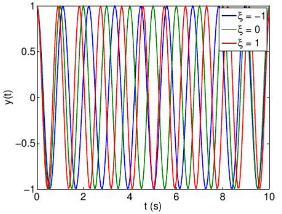
Quadrature - Projection $w^{(k)} \neq 1$



$$\forall j \in J, a_j = \sum_{k=1}^{K} Y^{(k)} H_j(\xi^{(k)}) w^{(k)}$$

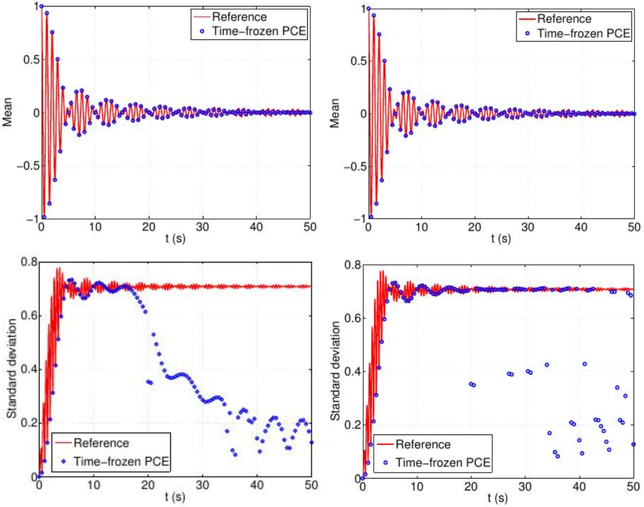
THE LONG-TERM INTEGRATION PROBLEM

- Classical gPC fails in representing very simple stochastic dynamics.
- $Y = \cos(\sqrt{k_0 + k_1 \xi}\, t)$
  - $k_0 = (2\pi)^2$ and $k_1 = 0.2 k_0$
  - $\xi \sim \mathscr{U}[-11]$



Source : Chu Van Mai PhD Thesis, 2016.

Up to degree 10 | Up to degree 30



A QUICK REVIEW

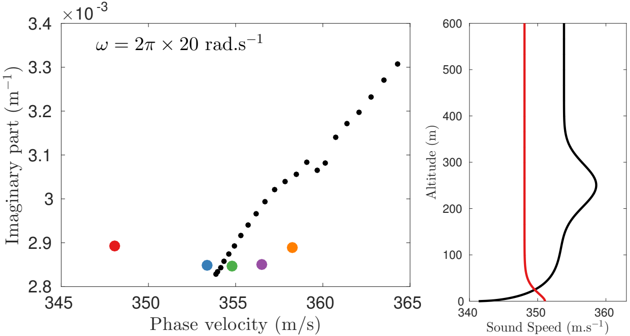
Existing methods for dealing with stochastic dynamics can be classified in two families :

- Work on the gPC :
  - Use high degree polynomials [Lucor and Karniadakis, 2004 ; Blatman and Sudret 2010], using techniques to limit the size of the basis [Jakeman et al., 2015 ; Hampton and Doostan, 2015 ; Doostan, 2013].
  - Multi-element approach to partition the input space [Wan and Karniadakis, 2005 ; Paffrath and Wever 2007] or the physical space [Chen et al. 2015].
  - Enrich the basis using non linear functions [Ghosh and Ghanem, 2008 ; Gosh and Iaccarino, 2007 ; Pettit and Beran, 2006 ; Le Maître et al., 2007].
- Try to capture the dynamics :
  - In the case of periodic systems [Witteveen and Bijl, 2008 ; Le Maître et al., 2010]
  - Use time-dependant polynomials [Gerritsma et al., 2010 ; Heuveline and Schick, 2014 ; Luchtenburg et al., 2014] or use a stochastic time [Mai et al., 2017]
  - Couple gPC with autoregressive processes [Spiridonakos and Chatzi, 2015 ; Wagner and Ferris, 2007 ; Kopsaftopoulos and Fassois, 2013 ; Samara et al., 2013 ; Sakellariou and Fassois, 2016, Mai et al., 2016]

For wave propagation in an inhomogeneous random medium, the dynamics can be characterized by the behaviour of the **spectral elements of the propagating operator**.

Outline

PROPAGATION MODEL : THE WAVE EQUATION



- Guided waves : $\left[\frac{d^2}{dz^2} + \frac{\omega^2}{c(z)^2}\right]\Psi(\omega,z) = k(\omega)^2\Psi(\omega,z)$
- On a bounded domain, the spectrum is only discrete but on a semi-open domain : $\sigma(H) = \sigma_{\text{disc}}(H) \oplus \sigma_{\text{ess}}(H)$.
- Only discrete modes (colored dots) have an acoustic contribution.
- Solution G of $\Delta G + \frac{\omega^2}{c(z)^2} G = 0$ can be decomposed on the eigenfunctions thanks to spectral theorem.

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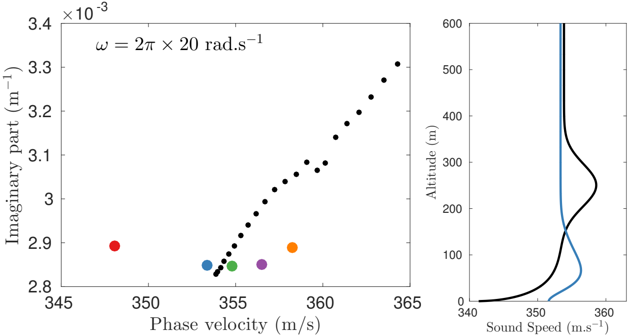

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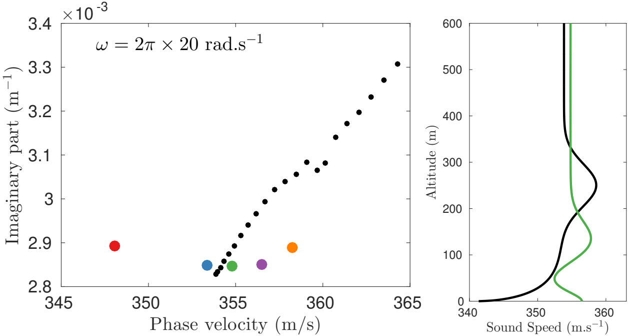

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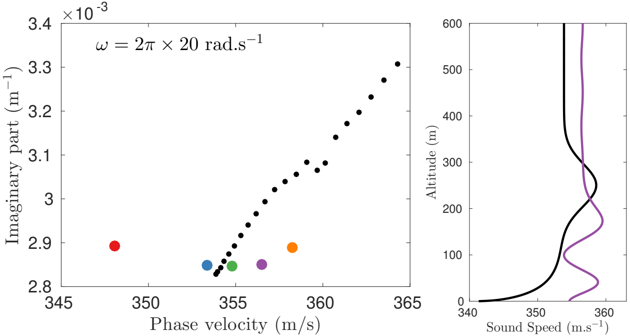

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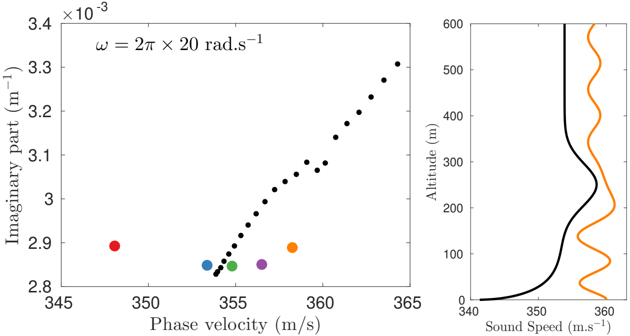

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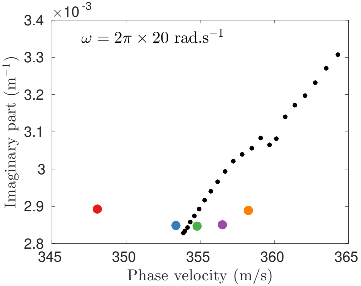
THE ACOUSTIC MODES

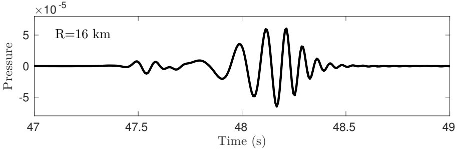
- Green function at distance R.

$$G(\omega,R) \sim \sum_{l=1}^{N} \underbrace{\frac{\Psi_l(\omega)^2 e^{ik_l(\omega)R}}{\sqrt{k_l(\omega)R}}}_{G_l(R,\omega)}$$

  - The number of discrete modes depends on frequency : $N = N(\omega)$.



- Signals and wavepackets.



$$\mathscr{F}^{-1}\left[\sum_{l=1}^{N_0} G_l s_0\right] = \sum_{l=1}^{N_0} \mathscr{F}^{-1}\left[G_l s_0\right]$$

$$N_0 = \max N(\omega)$$

- Random medium : $c(z)$ gives $(k_l, \Psi_l(z))$.
- A gPC metamodel for $Y = (k_l, \Psi_l(z))_{l=1,\dots,N}$ allows to compute the pressure field with a Fourier transform.

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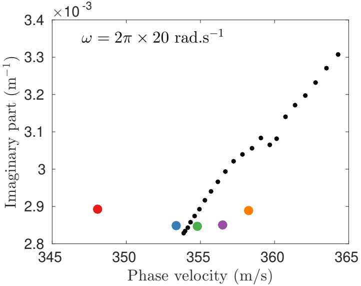

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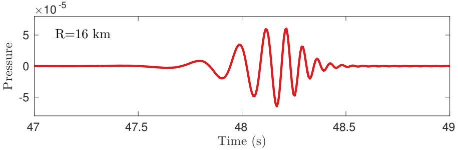

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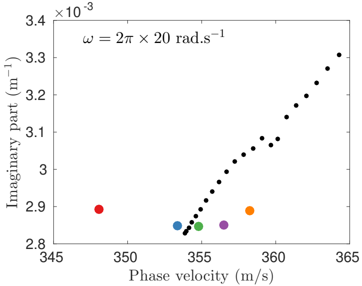

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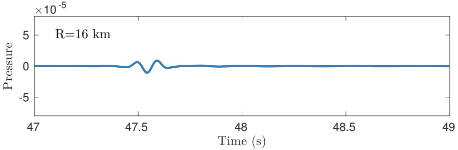

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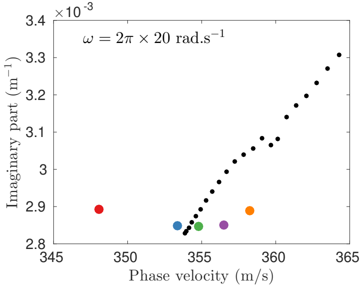

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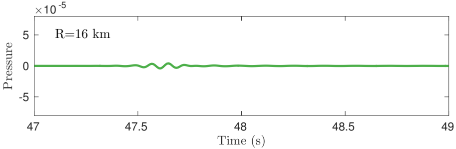

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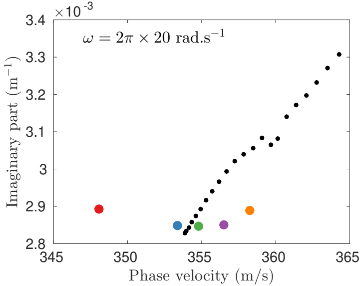

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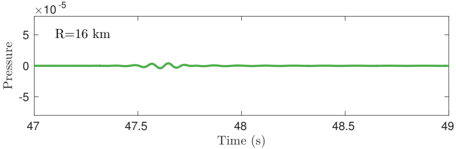

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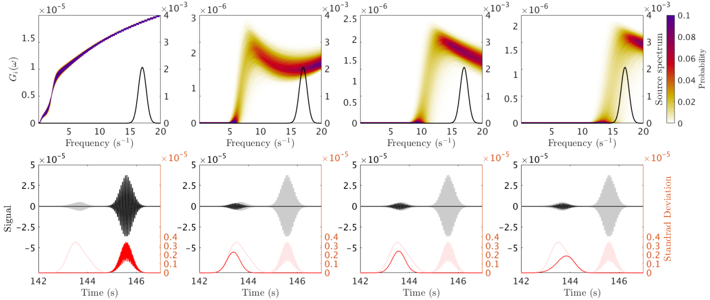
OUTLINE

gPC development of the spectral elements

- PDF of $G_i(\omega, \xi)$ as a function of ξ depends on the mode number i.
- Source is fixed and deterministic, we only look at variability generated by the random medium.

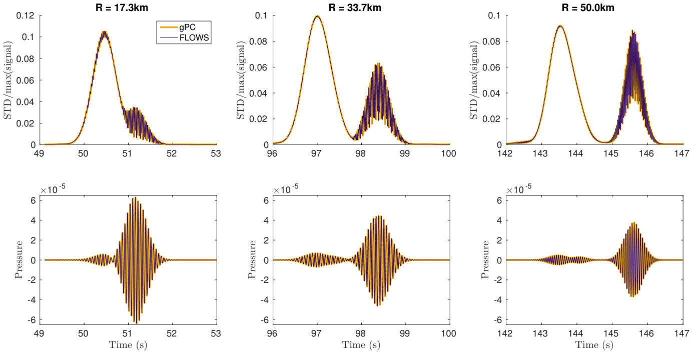


- gPC expansion for $Y = (k_l(\xi), \Psi_l(z,\xi))_{l=1,\dots,N}$:
  - $\tilde{k}_l(\xi) = \sum_{j \in J} a_j^{k_l} H_j(\xi)$
  - $\tilde{\Psi}_l(z,\xi) = \sum_{j \in J} a_j^{\Psi_l}(z) H_j(\xi)$

$\Rightarrow$ Metamodel for $s(t)$.

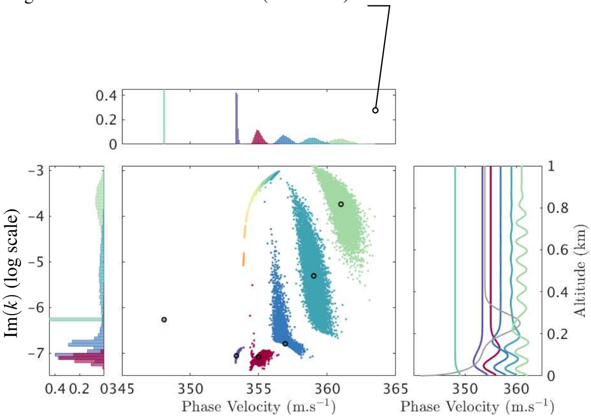
COMPARISON WITH MONTE-CARLO SIMULATIONS

- About convergence of signals $\tilde{s}(t,\xi)$ produced by the gPC-metamodel.
  - (1) gPC provides convergence in L^2-norm and (2) G depends continuously on k_l and Ψ_l. Hence $||\tilde{G}-G||_2 \xrightarrow{P\to\infty} 0$.
  - $\mathscr{F}$ is an L^2-isometry and thus, $||\tilde{s}(t,\xi)-s(t,\xi)||_2 \xrightarrow{P\to\infty} 0$.
- Comparison of gPC and Monte-Carlo simulations :



TRACKING MODES

Eigenvalues as random variables (ω is fixed)



Modes are function of ξ and ω but the dependance is unknown.

- Need to track the eigenvalue as a function of ξ to **isolate the QoI** and calibrate the metamodel for a fixed frequency.
- Need to track the eigenvalue as a function of ω to **compute the signal** associated with this mode for a fixed input parameter.

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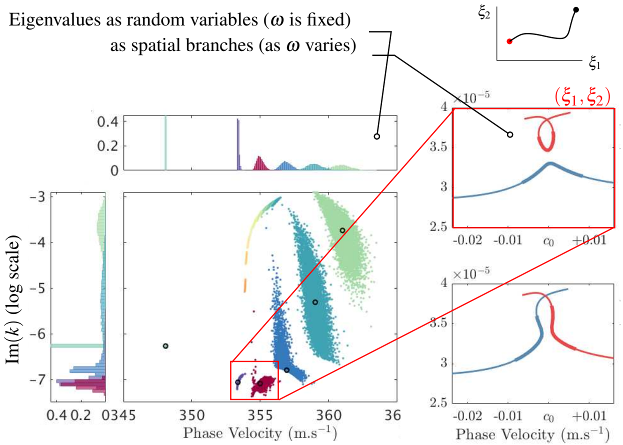

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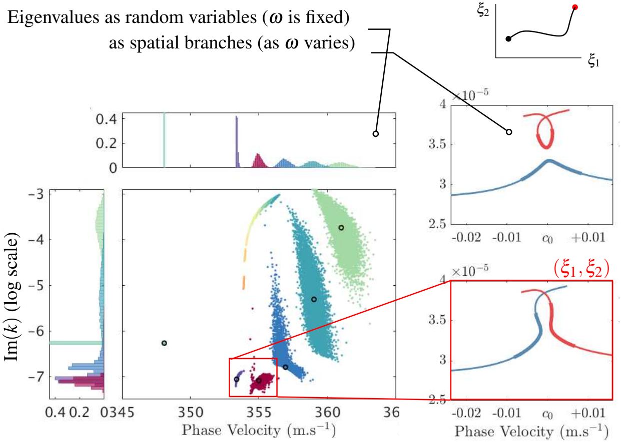

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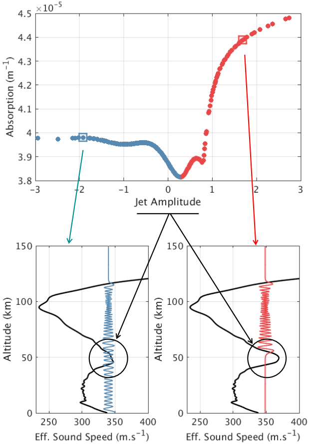
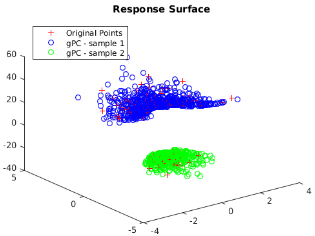
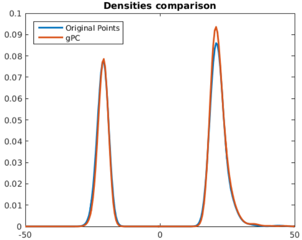
MODE SWITCHING : GPC FOR PARTIAL RESPONSE SURFACE



- A fixed mode can behave very differently depending on the profiles.
- Those regimes can be seperated in the parameter space in order to calibrate different gPC and capture the different physics :

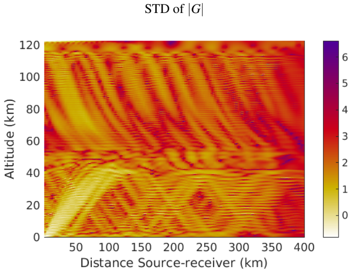




CONCLUSION AND PERSPECTIVES

- Take-home messages :
  - Computing gPC expansions on spectral elements allows to circumvent the long-term integration problem for wave propagation.
  - Our metamodel can be used for **any source** and at **any distance** which opens the way to bayesian inversion.
  - The spectral decomposition captures local perturbations and can be used for statistical model reduction.

- Perspectives :
  - Statistical characterization of the perturbation : Gravity Waves model.
  - Multi-scale approach to take small strucutures into account.
  - Write a manuscript . . .



A perturbative approach for small scale fluctuation

- Atmospheric perturbations include large deviation and turbulent noise :

$$c(z,\xi) = \underbrace{c_0(z,\xi)}_{\text{large}} + \underbrace{c_1(z)}_{\text{small}} = c_0(1+\varepsilon\mu).$$

- The impact of small-variance turbulence is modelled using the coupling matrix $(C_{nm})_{n,m}$

$$p \sim \sum_n G_n(\xi)\left[1 + i\sqrt{\varepsilon}R\frac{\omega^2 C_{nn}(\xi)}{2k_{n0}(\xi)}\right] + \sum_{n,m} D_{nm}(\xi)$$

where D_{nm} is obtained from C_{nm}.

- gPC expansion of C_{nm} (which depends on c_0) can be derived from $\phi_{n0} = \sum_{k=0}^{+\infty} \alpha_k^{(n)}(z)H_k(\xi)$:

$$\boxed{C_{nl} = \sum_{k=0}^{+\infty} \gamma_k^{(nl)} H_k(\xi)} \text{ where } \gamma_p^{(nl)} = \sum_{j,k}\left\langle \frac{\mu(z)\alpha_j^{(n)}(z)}{c_0^2(z)}, \alpha_k^{(l)}(z)\right\rangle \mathbb{E}\left[H_j H_k H_p\right]$$