DE LA RECHERCHE À L'INDUSTRIE

A polynomial chaos-based approach of infrasound propagation

17th International Symposium on Long Range Sound Propagation

A. Goupy [1,3], C. Millet[1,3] & D. Lucor[2]

[1] CEA, DAM, DIF, F-91297 Arpajon, France
[2] LIMSI-CNRS, F-91403 Orsay, France
[3] CMLA, ENS Paris-Saclay, F-94235 Cachan, France

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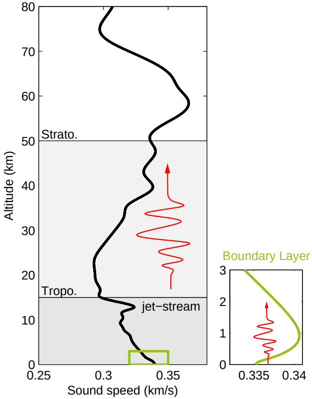
- On the use of Atmospheric Specifications (AS).
  - AS are provided by numerical weather forecasts / atmospheric climate reanalysis.
  - AS capture most efficient ducts but fail in representing small-scale fluctuations.
- The impredicable component of AS.
  - Epistemic uncertainty: lack of knowledge that can be reduced... Provided we can improve our physics' knowledge.
  - Aleatoric uncertainty: due to the natural variability of the propagation medium.
- Assessing the impact of those uncertainties:



« Knowledge increases by taking into account uncertainty, not by exorcising it. »

Edgar Morin, philosopher

Motivations

- On the use of Atmospheric Specifications (AS).
  - AS are provided by numerical weather forecasts / atmospheric climate reanalysis.
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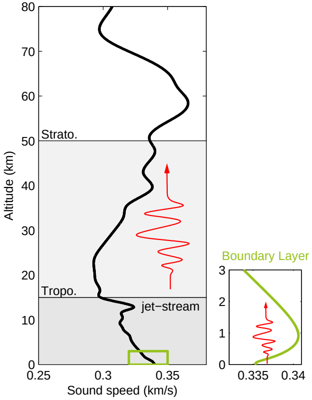


- The impredicable component of AS.
  - Epistemic uncertainty: lack of knowledge that can be reduced... Provided we can improve our physics' knowledge.
  - Aleatoric uncertainty: due to the natural variability of the propagation medium.
- Assessing the impact of those uncertainties:
  - The statistical approaches (Monte Carlo) need a large number of runs.
  - For fixed model's parameters θ, can we reduce the numerical cost by using a metamodel?

$$Y = F(X(\xi);\theta)$$

Outline

Polynomial Chaos decomposition

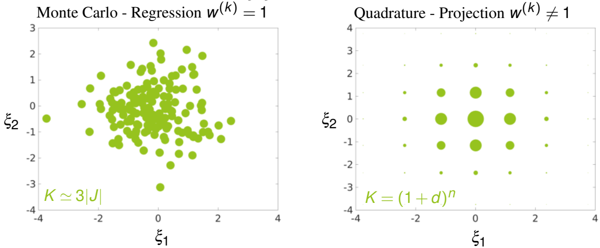
A non-intrusive metamodel of $Y = F(X(\xi);\theta)$ where $\xi \sim \mathcal{N}(0, I_n)$.

- Polynomial chaos decomposition*.

$$Y = \sum_{j\in J} a_j H_j(\xi) \text{ and } a_j = \langle Y, H_j \rangle,$$

  where $(H_j)_{j\in J}$ set of polynomials (up to degree d) that are orthonormal for scalar product $\langle f, g\rangle = \mathbb{E}[fg]$.

- Computation of coefficients $(a_j)_{j\in J}$.



$$a_j = \sum_{k=1}^{K} Y^{(k)} H_j(\xi^{(k)}) w^{(k)}.$$

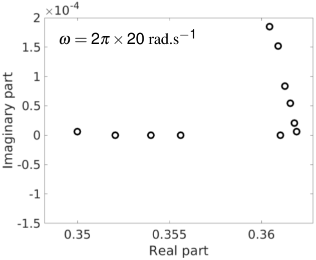
* Wiener 1938, Cameron & Martin 1947, Ghanem & Spanos 1991.

- A gPC model for $Y = (k_l, \Psi_l)_{l=1,\dots,N}$, where k_l and Ψ_l are eigenvalues and eigenfunctions (at ground level) of the Helmholtz equation.

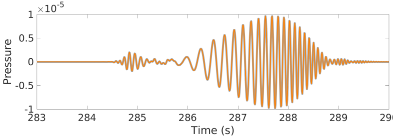
- Green function at distance R.

$$G(\omega, R, \xi) \sim \sum_{l=1}^{N} \underbrace{\frac{\Psi_l(\omega,\xi)^2 e^{ik_l(\omega,\xi)R}}{\sqrt{k_l(\omega,\xi)R}}}_{G_l(R,\omega,\xi)}$$

The number of modes depends on frequency and ξ: $N = N(\omega, \xi)$.



- Signals and wavepackets.



$$\mathscr{F}^{-1}\left[\sum_{l=1}^{N_0} \overline{G_l} s_0\right] = \sum_{l=1}^{N_0} \mathscr{F}^{-1}\left[\overline{G_l} s_0\right]$$

$\overline{G_l} = G_l 1_{\{\omega > \omega_l\}}$, ω_l: cut-off freq.

$N_0 = \max N(\omega; \xi)$

The propagation model

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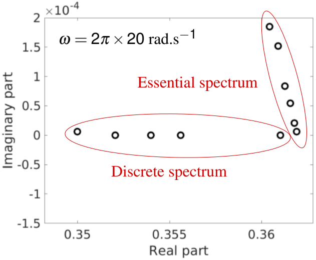

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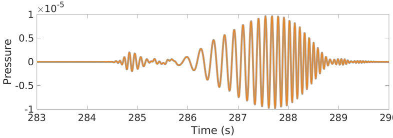

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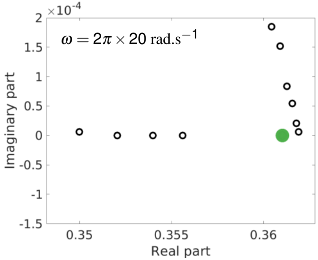

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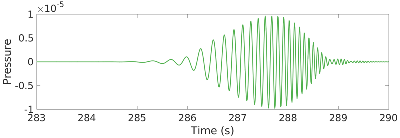

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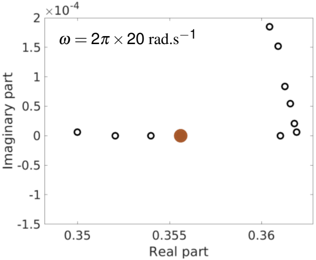

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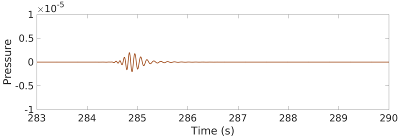

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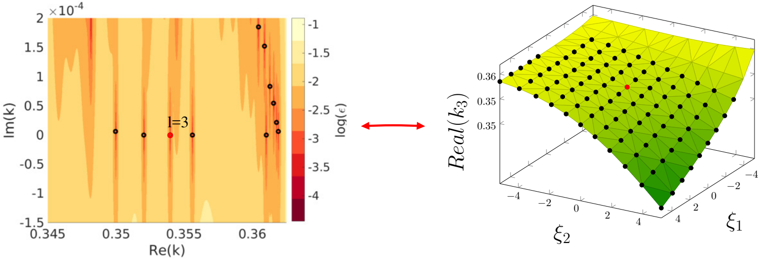
Computation of the gPC coefficients

- For a fixed frequency $\omega_0 = 2\pi \times 20$ rad.s^{-1} we are interested in the variability of a given eigenvalue:



Where pseudospectrum is defined by: $Sp_\varepsilon(A) = \{z \in \mathbb{C} / z \in Sp(A+E)$ with $||E|| < \varepsilon\}$

- The cut-off frequency is also developped on the gPC basis $\tilde{\omega}_l(\xi)$ and used to bound the domain of validity of the metamodel.
- Our metamodel has been computed using a regression on a quadrature grid, the optimal polynomial order is selected using a cross-validation technique.

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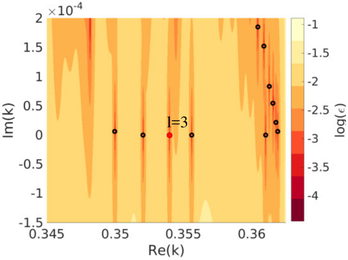

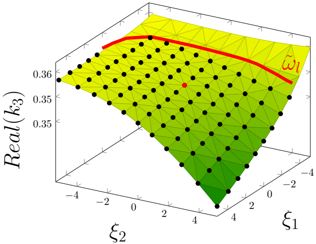

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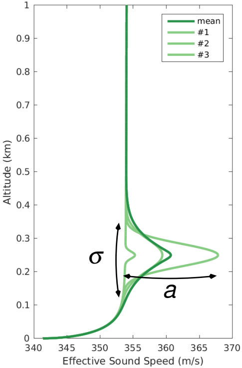
- A toy model: the Planetary Boundary Layer (PBL).
  - PBL + nocturnal jet*:
    PBL model from Waxler 2008
    $u_J(z,\xi) = a \times e^{\frac{z-z_J}{\sigma^2}}$ the nocturnal jet.
    $\Rightarrow$ Effective celerity approximation: $c(z,\xi)$
  - Uncertainties on the jet properties:
    - Jet amplitude: $a \sim N(m_a, s_a)$.
    - Jet spread: $\sigma \sim N(m_\sigma, s_\sigma)$.
    $\Rightarrow \xi = (a, \sigma)$



- Numerical setup.
  - Perfectly Matched Layer used as $z \to \infty$.
  - Neumann homogeneous condition at the ground.
  - Variance of the parameters $\to \sim$ 7% of fluctuation on the profil.

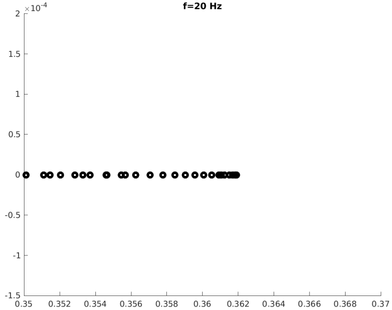
* Waxler *et al.*, JASA, 2008; Chunchuzov *et al.*, JASA, 1990, 2005, Wilson *et al.*, JASA, 2015.

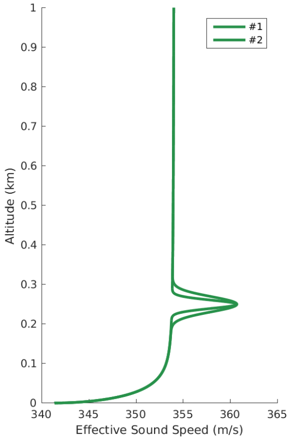
Technical issue: eigenvalue tracking

- With the classical Helmoltz operator:

$$H = \frac{\partial^2 \Psi}{\partial z^2} - \frac{\omega^2}{c(z)^2}\Psi$$

- Spectrum of H:





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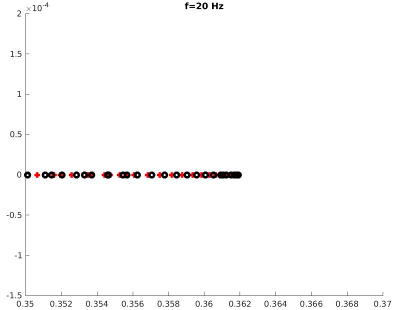

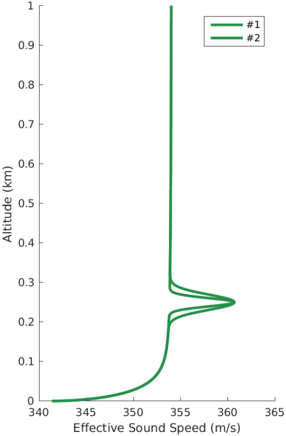

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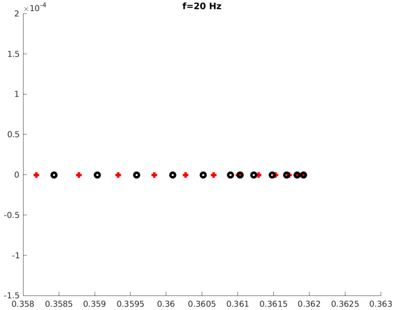

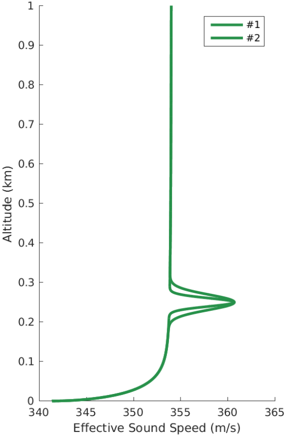

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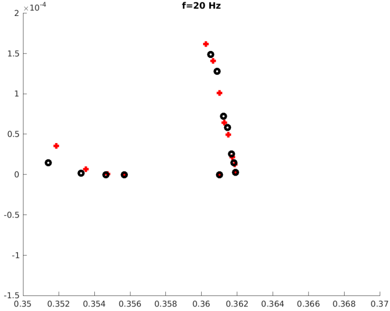
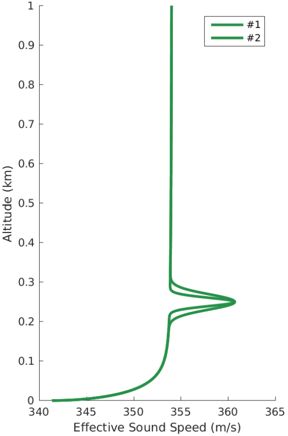
Technical issue: eigenvalue tracking

- Using a PML* $z \to \tilde{z} = z + if(z)$:

$$H = \frac{\partial^2 \Psi}{\partial \tilde{z}^2} - \frac{\omega^2}{c(\tilde{z})^2}\Psi$$

- Spectrum of H:





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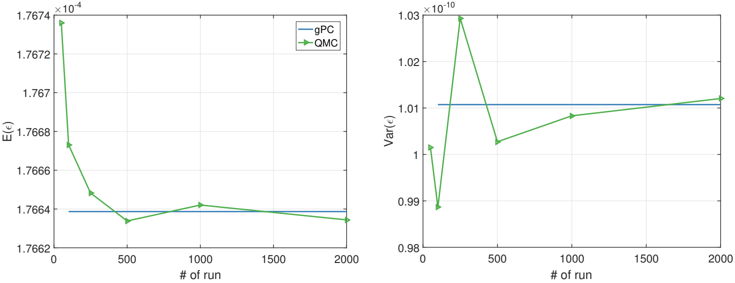
Convergence properties

- About convergence of signals $\tilde{s}(t,\xi)$ produced by the gPC-metamodel.
  - (1) gPC provides convergence* in L^2-norm and (2) G depends continuously on k_l and Ψ_l. Hence $||\tilde{G} - G||_2 \xrightarrow{P\to\infty} 0$.
  - $\mathscr{F}$ is an L^2-isometry and thus, $||\tilde{s}(t,\xi) - s(t,\xi)||_2 \xrightarrow{P\to\infty} 0$.
- Validation through a measure of discrepancy between gPC and QMC (Quasi Monte-Carlo).

$$\varepsilon = ||\tilde{s}(t,\xi) - s(t,\xi)||.$$

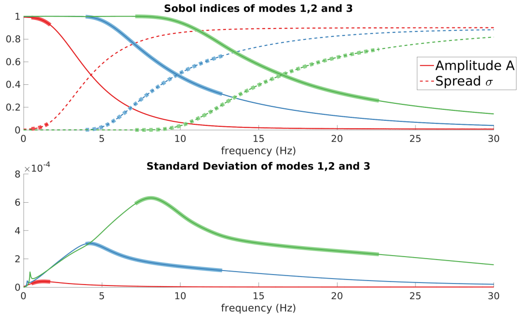


* Cameron Martin theorem.

- The Sobol index $S_r(k_l)$ measures the sensitivity of eigenvalue k_l to the parameter ξ_r:

$$S_r(k_l) = \frac{\mathrm{Var}(\mathbb{E}[k_l|\xi_r])}{\mathrm{Var}(k_l)}$$

- gPC allows direct computation of Sobol indices so as to assess the role of each input (random) parameter - here a and σ, onto $\mathrm{Var}(k_l)$ ($l \leq N$).



Conclusion

- Efficient way to obtain a metamodel for complex signals, provided the dimension of stochastic inputs ξ is rather small (typically less than 10).
- The metamodel can be used to produce signal statistics and sensitivity indices at low cost, thereby allowing its intensive use in operational-like environments.
- Main limitation comes from the ability to track your random eigenvalues in the complex plane.
- Short-term development: take into account uncertainties on the source (for the next LRSP Symposium ☺!).

Thank you for your attention !