Polynomial chaos expansion for acoustic propagation

Séminaire des doctorants du CMLA

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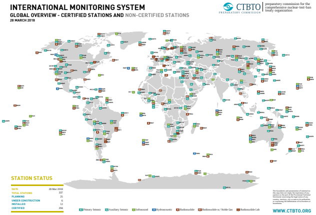
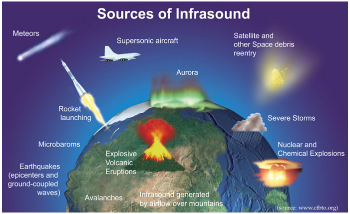
Context

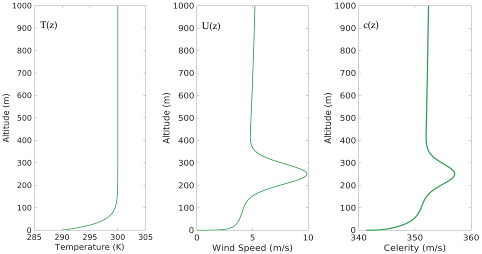


- ▶ The verification regime of the Comprehensive Nuclear-Test-Ban Treaty (CTBT) is designed to detect any nuclear explosion conducted on Earth – underground, underwater or in the atmosphere.
- ▶ Infrasound monitoring is one of the four technologies used by the International Monitoring System (IMS) to verify compliance with the CTBT.



- ▶ Infrasound has the ability to cover long distances with little dissipation.
- ▶ Infrasound signals can be severly distorted by the propagation in the atmosphere.

Planetary Boundary Layer



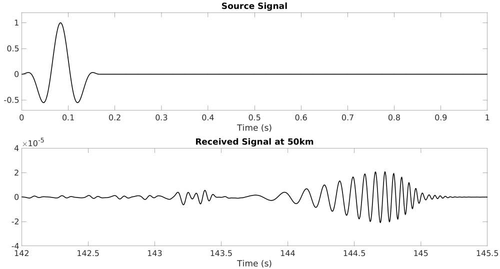
Wave celerity:

$$c(z) = \sqrt{\gamma R T(z)} + U(z)$$

where:

- $T(z)$ is the temperature profile
- $U(z)$ the wind profile.

▶ The celerity profile characterizes the medium for the propagation.



NORMAL MODES (1/2)

- $(k_i(\omega), \Psi_i(\omega, z))$ eigenvalues and eigenfunctions of H:

$$H\Psi = \frac{\partial^2 \Psi}{\partial z^2} + \frac{\omega^2}{c(z)^2}\Psi = k\Psi$$

- Green function:

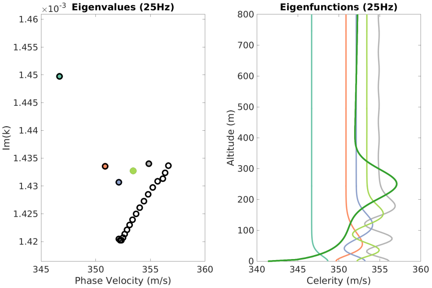
$$G(\omega) = \sum_{i=1}^{N} G_i(\omega) = \sum_{i=1}^{N} \alpha \frac{\Psi_i(\omega, 0)^2}{\sqrt{k_i(\omega)R}} e^{ik_i(\omega)R}$$

with $\alpha = \frac{e^{-i\pi/4}}{\sqrt{8\pi}}$

R: distance source-receiver.

- Signal:

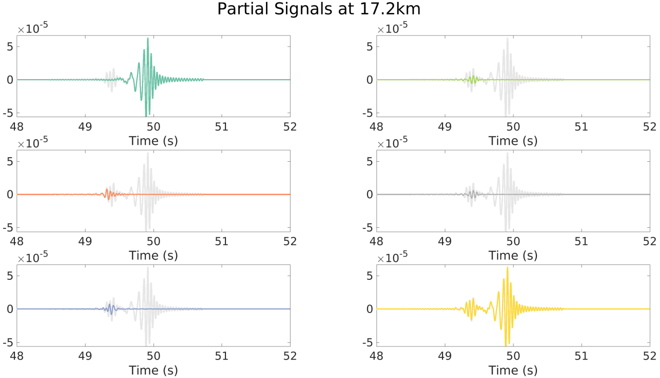
$$p(t) = \mathcal{F}^{-1}[G(\omega)s(\omega)](t)$$



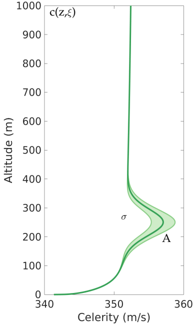
Normal Modes (2/2)

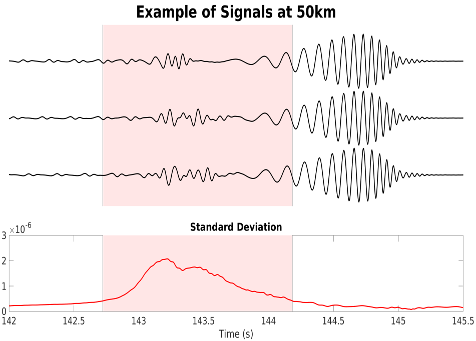
Signal reconstructed with only one mode:

$$p_i(t) = \mathcal{F}^{-1}[G_i(\omega)s(\omega)](t)$$



Random Jet





Impact of the uncertainties on the medium on the acoustic signal received at the ground?

▶ Metamodel of **the eigenpairs of the propagation operator** able to generate such signals.

POLYNOMIAL CHAOS DECOMPOSITION (1/2)

Build a metamodel of $X = \mathcal{M}(\boldsymbol{\xi})$ where $\boldsymbol{\xi} \sim \mathcal{N}(0, \mathbb{I}_n)$:

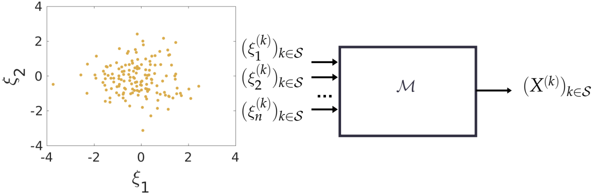
Polynomial chaos decomposition:

- $(H_j)_{j \in J}$ set of orthonormal polynomials for $\langle f, g \rangle = \mathbb{E}[fg]$
- $X(\boldsymbol{\xi}) = \sum_{j \in J} a_j H_j(\boldsymbol{\xi})$ where $a_j = \langle X, H_j \rangle$
- taking polynomials up to degree d, $|J| = \frac{(n+d)!}{n!d!}$

POLYNOMIAL CHAOS DECOMPOSITION (2/2)

Coefficients $(a_j)_{j\in J}$ can be computed:

- with Monte-Carlo using a sample $\mathcal{S}$:

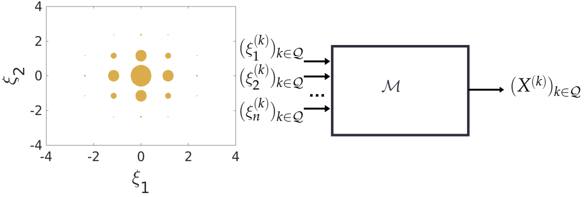


$$a_j = \sum_{k\in\mathcal{S}} X^{(k)} H_j(\boldsymbol{\xi}^{(k)})$$

▶ $|\mathcal{S}| \approx 3\frac{(n+d)!}{n!d!}$

$n = 2, d = 10 : |\mathcal{S}| \approx 200$

- with a quadrature $\mathcal{Q}$:



$$a_j = \sum_{k\in\mathcal{Q}} X^{(k)} H_j(\boldsymbol{\xi}^{(k)}) w^{(k)}$$

▶ $|\mathcal{Q}| = (d+1)^n$

$n = 2, d = 10 : |\mathcal{Q}| = 121$

gPC decomposition of the acoustic modes

- gPC decomposition of each eigenvalue and eigenvector at the ground

$$\widehat{k_i}(\omega, \boldsymbol{\xi}) = \sum_{j \in J} a_j^{k_i}(\omega) H_j(\boldsymbol{\xi}) \text{ and } \widehat{\Psi_i}(\omega, \boldsymbol{\xi}) = \sum_{j \in J} a_j^{\Psi_i}(\omega) H_j(\boldsymbol{\xi})$$

- Reconstruction of the Green function

$$\widehat{G}(\omega, \boldsymbol{\xi}) = \sum_{i=1}^{N} \widehat{G}_i(\omega, \boldsymbol{\xi}) = \sum_{i=1}^{N} \alpha \frac{\widehat{\Psi_i}(\omega, \boldsymbol{\xi})^2}{\sqrt{\widehat{k_i}(\omega, \boldsymbol{\xi}) R}} e^{i\widehat{k_i}(\omega, \boldsymbol{\xi}) R}$$

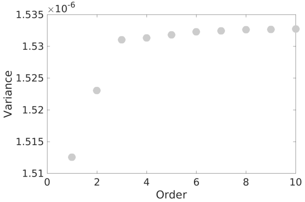
- Simulation of signals using the metamodel:

$$p(t, \boldsymbol{\xi}) = \mathcal{F}^{-1}[\widehat{G}(\omega, \boldsymbol{\xi}) s(\omega)](t)$$

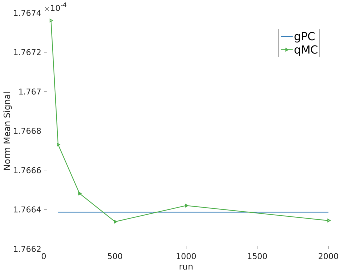
Convergence

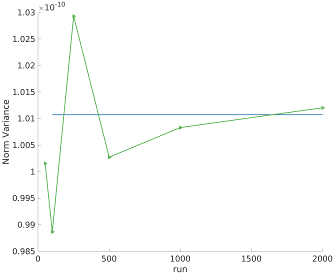
- Variance can be used to control the convergence:

$$Var[k_5] = \sum_{j \in J \setminus \{0\}} (a_j^{k_5})^2$$



- Statistics on the signals can be compared with those obtained by Monte-Carlo simulations:



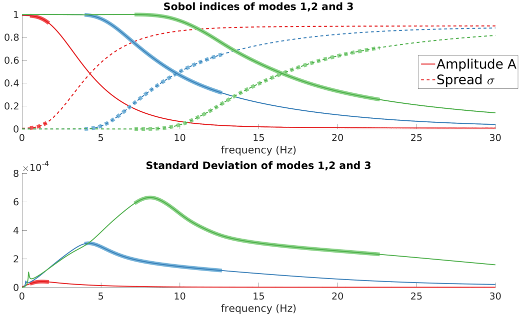


Sensitivity Analysis

- Sobol index $S_j(k_i)$ gives a measure of the sensibility of mode k_i to the parameters ξ_j:

$$S_j(k_i) = \frac{Var(\mathbb{E}[k_i|\xi_j])}{Var(k_i)}$$

- The gPC decomposition allows a quick computation of the Sobol indices: $S_j(k_i) = \sum_{j \in J'} (a_j^{k_i})^2$ where $J' = \{k \in J^* | \exists Q \in \mathbb{R}[X], P_k(\boldsymbol{\xi}) = Q(\xi_j)\}$



Conclusion:

- gPC representation of the acoustic modes
- Metamodel able to give statistics on the signals
- Sensitivity analysis on the modes

Perspectives:

- Use model reduction to select modes with high acoustical contribution and great sensitivity to the uncertainties.
- Develop an eigenvalue tracking method to generalize to a realistic atmosphere.